Thesis

Riemann's P-function.

by

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Dissertation
presented for the degree
of
Doctor of Philosophy
in the
Johns Hopkins University.

1890.

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Introduction.

The following paper was begun at the suggestion of Dr. Craig and has had the benefit of his advice and criticism. It is an attempt to set forth in a clear and readable form the properties of the P-function invented by Riemann and treated of in his collected works.

Part I of the present paper is devoted to a restatement in systematic order of the properties of the function as given by Riemann, with brief demonstrations of certain points at which he has only hinted and original proofs of some propositions which he considered almost self-evident, but which admit, and perhaps require, elaborate demonstrations.

Section 4 of Part I is an instance of this kind; the work therein and the table of equivalent functions are my own. The same

may be said of section 3, where, by using an expansion in powers of the variable which is known to be convergent in the proper regions, the ideas are fixed and a certain rule of statement is attained.

In proving that the P-function satisfies a linear differential equation of the second order, I have followed Riemann except that certain points in the theory of functions have been touched more lightly, as being better known to readers at the present day.

Part II is entirely my own. It is a study of the differential equation satisfied by the P-function from the point of view of the modern theory as originated by Fuchs and developed by his illustrious collaborators. The theorems concerning the exponents are stated and proved as properties of the indicial equation in Section 4. Section 6 is an elaborate study of the transformation $x' = \frac{ex+f}{gx+h}$; Section 5 is devoted to obtaining the coefficients of the differential equation and in subsequent sections the speci-

cal Harmonics, Toroidal Functions and Bessel's Functions are expressed as P functions, while, in conclusion, the P function itself is expressed as a Hypergeometric Series.

P A R T 1.

Section 1. Definition of the P-function.

Conceive that a function exists, which has the following properties :-

1. It is finite and continuous throughout the plane of imaginary quantities, except at the points $x = a$, $x = b$, $x = c$

2. Between any three branches of the function, P', P'', P''', there exists a linear relation with constant coefficients,

$$c'P' - c''P'' + c'''P''' = 0.$$

3. The function may be put in any of the forms

$$C_\alpha P^{(\alpha)} + C_{\alpha'} P^{(\alpha')}, \quad C_\beta P^{(\beta)} + C_{\beta'} P^{(\beta')}, \quad C_\gamma P^{(\gamma)} + C_{\gamma'} P^{(\gamma')}$$

Where C_α, $C_{\alpha'}$, C_β, $C_{\beta'}$, C_γ, $C_{\gamma'}$ are constants ; and the expressions $P^{(\alpha)}(x-a)^{-\alpha}$, $P^{(\alpha')}(x-a)^{-\alpha'}$

become neither zero nor infinite when $x = a$; likewise $P^{(\beta)}(x-b)^{-\beta}$, $P^{(\beta')}(x-b)^{-\beta'}$ are neither zero nor infinite for $x = b$ and $P^{(\gamma)}(x-c)^{-\gamma}$, $P^{(\gamma')}(x-c)^{-\gamma'}$ are neither zero nor infinite for $x = c$.

By these properties, the P-function is completely defined, except that it contains two arbitrary constants. It is designated by the symbol $P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma, & x \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$

Section 2. The quantities $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$.

These quantities may be anything whatever subject to the conditions ;

1. None of the differences $\alpha - \alpha'$, $\beta - \beta'$, $\gamma - \gamma'$ shall be an integer.

2. The sum of all the quantities is constantly unity, *i.e.* $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$

Section 3. Properties of the P - function.

1. The first three vertical columns may be interchanged at pleasure. For, when the defining conditions are applied to the three functions so obtained, no distinction can be observed between them; hence they are identical, provided the conditions actually define a function.

2. In the same way, we see that α may be interchanged with α', β with β', and γ with γ'.

3. Let X be replaced by X', a rational linear function of X, so taken that when

$$X = a,\ X' = a'$$
$$X = b,\ X' = b'$$
$$X = c,\ X' = c'$$

then the two functions $P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$ and $P\begin{Bmatrix} a', & b', & c' & \\ \alpha, & \beta, & \gamma & x' \\ \alpha', & \beta', & \gamma' & \end{Bmatrix}$

are equal. By this transformation, to be fully developed later on, every ~~per~~ P-function may be expressed in terms of another, whose singular points are $0, \infty, 1$. But every function having the same $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ may thus be put in the form $P\begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$, and our definition will then make no distinction between them. That is: ~~all~~ All P - functions having the same exponents, $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ may be reduced to the same function $P\begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$ which may be briefly written $P\begin{Bmatrix} \alpha & \beta & \gamma & \\ \alpha' & \beta' & \gamma' & x \end{Bmatrix}$

According to the linear expression in x which we choose for the variable, the points $0, \infty, 1$ may appear in six different ways, corresponding to six modes of propagation of the function in the plane of x. They are -

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};\quad P\begin{Bmatrix} \infty & 0 & 1 & \\ \alpha & \beta & \gamma & \frac{1}{x} \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};\quad P\begin{Bmatrix} 1 & \infty & 0 & \\ \alpha & \beta & \gamma & 1-x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha & \beta & \gamma & \frac{x}{x-1} \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};\quad P\begin{Bmatrix} \infty & 1 & 0 & \\ \alpha & \beta & \gamma & \frac{x-1}{x} \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};\quad P\begin{Bmatrix} 1 & 0 & \infty & \\ \alpha & \beta & \gamma & \frac{1}{1-x} \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

Section 4. Transformation of Exponents.

We shall have, by definition, the product $P^{(\alpha\ \delta)}(x-a)^{-\alpha}$ neither 0 nor ∞ for $x = a$; hence, consistently with all that precedes we may write, denoting by P_1 a new P function,

$P_1^{(\alpha+\delta)} = (x-a)^{\delta} P^{(\alpha)} (x-b)^{-\delta}$, if we choose; and
$P_1^{(\alpha'+\delta)} = (x-a)^{\delta} P^{(\alpha')} (x-b)^{-\delta}$; $P_1^{(\beta-\delta)} = (x-b)^{-\delta} P^{(\beta)} (x-a)^{\delta}$;
$P_1^{(\beta'-\delta)} = (x-b)^{-\delta} P^{(\beta')} (x-a)^{\delta}$; $P_1^{(\gamma)} = (x-a)^{\delta} (x-b)^{-\delta} P^{(\gamma)}$;
$P_1^{(\gamma')} = (x-a)^{\delta} (x-b)^{-\delta} P^{(\gamma')}$.

Observing that the left hand members of these equations are the [illegible] of the [illegible] $P_1 \begin{Bmatrix} a & b & c & \\ \alpha+\delta & \beta-\delta & \gamma & x \\ \alpha'+\delta & \beta'-\delta & \gamma' & \end{Bmatrix}$, we have the relation

$$P_1 \begin{Bmatrix} a & b & c & \\ \alpha+\delta & \beta-\delta & \gamma & x \\ \alpha'+\delta & \beta'-\delta & \gamma' & \end{Bmatrix} = \left(\frac{x-a}{x-b}\right)^{\delta} P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

If the P- function be in the reduced form

$$P \begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

then in the region of the point ∞ either branch has the form $\left(\frac{1}{x}\right)^{\beta}\left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots\right) = \left(\frac{1}{x}\right)^{\beta} y$; because $P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$ had in the region of the point b the form $(x-b)^{\beta}\left[a_0 + a_1(x-b) + a_2(x-b)^2 \dots\right]$.

and a transformation of such a nature that when $x = b$, x' shall become infinite, can only be of the form $x - b = \frac{1}{x'}$; hence the transformed function in the

region of the point ∞ has the form

$$x^{-\beta}\left[a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots\right] = x^{-\beta}\,\mathfrak{P}\left(\frac{1}{x}\right)$$

It follows, that, $P\{(1-x)^{\epsilon} x^{\delta} \dots\}$

will, in the region of the point ∞, be of the form

$$c_\beta \frac{(1-x)^{\epsilon} x^{\delta}}{x^{\beta}} \mathfrak{P}^{(\infty)} + c_{\beta'} \frac{(1-x)^{\epsilon} x^{\delta}}{x^{\beta'}} \mathfrak{P}^{(\infty')}$$ where $\mathfrak{P}$ and

$\mathfrak{P}^{(\infty')}$ are neither zero nor infinite for $x = \infty$. Clearly then, putting this in the form $c_\beta P^{(\infty)} + c_{\beta'} P^{(\infty')}$ we see that

$$P^{(\infty)} x^{\beta-\delta-\epsilon} \text{ and } P^{(\infty')} x^{\beta'-\delta-\epsilon}$$

are neither 0 nor ∞ at the point ∞; hence,

$$P\begin{Bmatrix} \alpha & \beta & \gamma & \\ \alpha' & \beta' & \gamma' & x \end{Bmatrix} (1-x)^{\epsilon} x^{\delta} = P\begin{Bmatrix} \alpha+\delta & \beta-\delta-\epsilon & \gamma+\epsilon & \\ \alpha'+\delta & \beta'-\delta-\epsilon & \gamma'+\epsilon & x \end{Bmatrix}$$

the first and last exponents being transformed by the rule found above.

Here ~~We see~~ δ and ϵ may have any values whatever, and this remark permits us to draw the following inference :

The values of any two of the exponents may be changed at pleasure, by introducing proper multipliers; but the sum $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'$ must remain unchanged, and always equal to 1. The differences $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ must also remain unaltered in absolute magnitude. In other words,

the product of a P - function by factors which fulfil the above conditions, may be expressed as a P - function.

Again, P - functions in which the differences $\alpha - \alpha'$, $\beta - \beta'$, $\gamma - \gamma'$ are the same, can differ only by determinate factors, as the following table will more fully illustrate. The transformations involved will be considered in Part II.

$$P\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix} = \begin{cases} (1-x)^{\gamma} x^{\alpha} P\begin{pmatrix} 0 & \beta+\alpha+\gamma & 0 & \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & x \end{pmatrix} \\ (1-x)^{\gamma'} x^{\alpha'} P\begin{pmatrix} \alpha-\alpha' & \beta+\alpha'+\gamma' & \gamma-\gamma' & \\ 0 & \beta'+\alpha'+\gamma' & 0 & x \end{pmatrix} \\ (1-x)^{\gamma} x^{\alpha'} P\begin{pmatrix} \alpha-\alpha' & \beta+\alpha'+\gamma & 0 & \\ 0 & \beta'+\alpha'+\gamma & \gamma'-\gamma & x \end{pmatrix} \\ (1-x)^{\gamma'} x^{\alpha} P\begin{pmatrix} 0 & \beta+\alpha+\gamma' & \gamma-\gamma' & \\ \alpha'-\alpha & \beta'+\alpha+\gamma' & 0 & x \end{pmatrix} \end{cases}$$

[illegible]

[illegible]

[illegible]

Section 5. Case where $\alpha' - \gamma' = \frac{1}{2}$; P

By section 4, this may always be reduced to the particular case $\gamma' = 0$, $\alpha' - \frac{1}{2}$. We shall then have

[illegible]

and the function is

$$P \left\{ \begin{matrix} \ & \ & 1 & \ \\ \frac{1}{2} & \ & & x \end{matrix} \right\}$$

In the region of the point $X = 0$ the function may by definition be put in the form

1) [illegible]

in the region of the point infinity,

2) [illegible]

and in the region of the point $X = 1$,

3) [illegible]

the series being convergent.

If now we write

$$x = z, \text{ or } x = z^2,$$

the series (1) will no longer have the point $z = 0$ for a branch point, since it will contain no fractional exponents.

The series (2) will become

[illegible]

and the transformed P - function will have the point [illegible] for a branch point with the exponents [illegible]

The series (3) becomes

[illegible]

for which the points $+1$ and -1 are now branch points, each with the exponents γ, γ'.

Hence the substitution $x = z^2$ yields a P - function whose branch points are $+1$, -1 and ∞, with the exponents $\gamma, \gamma'; \gamma, \gamma'; 2\beta, 2\beta'$, respectively ; and inasmuch as we have altered only the mode of propagation of the function, we may write

[illegible]

The sum of the exponents is now

[illegible]

If the difference [illegible] is an integer, the function

P [illegible]

no longer corresponds to the definition of a P - function,

and before effecting the transformation $\sqrt[3]{x-z}$ upon

P

we should so change the exponents

that this may not occur.

Section 6.

In the region of the point 0 the form of the function

is (1) c_1

In the region of the point ∞

(2)

and in the region of the point 1,

(3) $c_3 (x-1)$

If in these we make $X = Z$, then the point x = o

and x $= \infty$ are no longer branch points for 1) and 2) respectively while 3) becomes

ρ being a cube root of unity ~~which~~ This has for branch points

each with the exponents

Hence by the transformation X = Z we obtain

Here we must have

as it should.

Section 7. Applications of the preceding transformations.

Let the differences be denoted respectively by λ, μ, ν; and the function P by

$P(\lambda, \mu, \nu, x)$.

Since the values 0, ∞, 1 for $\frac{1-x}{1}$ correspond to the values 1, ∞, 0 for x, therefore, by Section 3)

$P(\mu, \nu, 2, 1, x) \quad P(\ldots, \nu, \ldots, x) = P\{\ldots\}$

by Section 5) Noting, that the values -1, ∞, 1 for $\bar{x}$ correspond to ... we obtain finally,

(1) ...

By these relations, P - functions which have two differences the same, or one difference equal to 1/2, are mutually expressible.

Again, observing that the values 0, ∞, 1 for x correspond to 1, 0, ∞ for $\frac{1}{1-x}$, it follows, that,

$P(\ldots, \nu, x) = P(\ldots) = P\{\ldots\}$

if $\nu - \ldots$ And, since the values 1

correspond to [illegible]

P [illegible]

Put [illegible] then because two differences are the same in P ([illegible]) we may apply equation (1) and thus obtain P $(\mathcal{V}, \mathcal{V}, \mathcal{V}, X,)$ = P [illegible]

$= P($ [illegible] $)$ [illegible] x_2.

And again

P [illegible]

[illegible]

Writing [illegible] we find that

P [illegible] $X_4)$ [illegible]

and if [illegible] $- X_5$ [illegible]

$- P$ [illegible] $= X_6$.

Thus we have

2) [illegible]

[illegible]

all mutually expressible.

Making find also

(3)

and making

(4)

The function

and writing this becomes

and by making X = it again transforms to

(We shall recur to this function in Part II.)

Section 8. Study of the Reduced function.

To study more completely the reduced function, let a cut be drawn from 1 to and from to o; so long as the cut is not crossed, P ( ) is uniform and continuous, and are uniform functions of X, when values are known, except constant factors depending on when P is known. By hypothesis

when are uniform in the region of the point o.

Hence if X makes a tour around the point 0 we shall have

$$+ C$$

The determinant

is not zero, since α is not an integer; and therefore P and P can be expressed linearly in terms of P,P'. The same is true of P , and P. It follows, that, can be expressed linearly, with constant coefficients in terms of or . Accordingly we make the assumptions

We may express these equations symbolically thus,

where denotes the substitution

, and from this,

If now, X makes a tour around the point we shall have, denoting by B the effect on

Since [illegible] is evidently commutative with [illegible]

Hence, separating symbols, we have for [illegible]

[illegible]

Likewise, if C denotes the effect upon [illegible] of a circuit around the point 1, we have

$$C = [illegible]$$

where [illegible]

Finally, if A denote the substitution caused in [illegible] by a tour around the point 0, we know that

[illegible]

since there are no other branch points, it follows that

$$1 = e^{2\pi i(\ldots)}$$

by forming the products of the determinants of the substitutions. This is consistent with the hypothesis that

[illegible]

We may now further investigate the quantities

[illegible]

as follows. -

A negative tour around the point 0, changes [illegible] to [illegible] but [illegible] ; hence this tour changes [illegible] to

Making now a negative tour around the point X, and denoting its effect on P by S, we have

[illegible]

But, the combined effect of the two tours is that of a positive tour around the point 1; therefore, since

[illegible] we have

[illegible] whence

1) [illegible]

In a manner precisely similar, we obtain the equation

2) [illegible]

Multiplying 1) by [illegible], σ being arbitrary,

[illegible]

From this subtract [illegible] multiplied by [illegible]

We thus obtain

[illegible]

Remembering that [illegible] this becomes, omitting the factor [illegible],

3) [illegible]

In like manner we obtain from

(4) [illegible]

We may so determine σ in each of equations (3) and (4) that one of the functions, say P^{γ} shall have its coefficient equal to 0.

To this end [illegible] must [illegible] whence [illegible] must $= 0, \pm 1, \pm 2, \ldots$

That is, σ must $= \gamma$ or differ from it by an integer only; choosing $\sigma = \gamma'$, equations (3) and (4) become

$$(5)\quad \alpha_\beta P^{\beta} e^{-\pi i(\alpha+\beta)} \sin(\alpha+\beta+\gamma)\pi + \alpha_{\beta'} P^{\beta'} e^{-\pi i(\alpha+\beta')} \sin(\alpha+\beta'+\gamma)\pi = \alpha_\gamma P^{\gamma'} e^{\pi i \gamma'} \sin(\gamma-\gamma')\pi$$

$$(6)\quad \alpha'_\beta P^{\beta} e^{-\pi i(\alpha'+\beta)} \sin(\alpha'+\beta+\gamma)\pi + \alpha'_{\beta'} P^{\beta'} e^{-\pi i(\alpha'+\beta')} \sin(\alpha'+\beta'+\gamma)\pi = \alpha'_\gamma P^{\gamma'} e^{\pi i \gamma'} \sin(\gamma-\gamma')\pi.$$

Eliminating $P^{\gamma'}$ we find at last the homogeneous equation in P^{β} and $P^{\beta'}$:

$$\frac{\alpha_\beta}{\alpha_\gamma} P^{\beta} e^{-\pi i(\alpha+\beta)} \sin(\alpha+\beta+\gamma)\pi + \frac{\alpha_{\beta'}}{\alpha_\gamma} P^{\beta'} e^{-\pi i(\alpha+\beta')} \sin(\alpha+\beta'+\gamma)\pi$$

$$= \frac{\alpha'_\beta}{\alpha'_\gamma} P^{\beta} e^{-\pi i(\alpha'+\beta)} \sin(\alpha'+\beta+\gamma)\pi + \frac{\alpha'_{\beta'}}{\alpha'_\gamma} P^{\beta'} e^{-\pi i(\alpha'+\beta')} \sin(\alpha'+\beta'+\gamma)\pi.$$

But since β is not equal to β', $\frac{P^{\beta}}{P^{\beta'}}$ cannot be a constant;

for [illegible]

being neither 0 nor [illegible] for X [illegible] which is the same as saying that each of them has a term independent of X.

Let [illegible] be these terms; then [illegible]

and [illegible]

which can not be a constant. [illegible]

Hence the coefficients of [illegible] and [illegible] must separately vanish, and the following relations result ;

$$7)\quad \frac{\alpha_\gamma}{\alpha'_\gamma} = \frac{\alpha_\beta}{\alpha_\beta}\,\frac{\sin(\alpha+\beta+\gamma')\pi}{\sin(\alpha+\beta+\gamma)\pi}\,e^{-\pi i(\alpha'-\alpha)} = \frac{\alpha_{\beta'}}{\alpha'_{\beta'}}\,\frac{\sin(\alpha'+\beta'+\gamma')\pi}{\sin(\alpha'+\beta'+\gamma)\pi}\,e^{\pi i(\alpha'-\alpha)}$$

Again, if we eliminate [illegible] we shall have [illegible] and as before.

or simply by interchanging [illegible] and [illegible] in 7)

$$8)\quad \frac{\alpha_{\gamma'}}{\alpha'_{\gamma'}} = \frac{\alpha_\beta}{\alpha'_\beta}\,\frac{\sin(\alpha+\beta+\gamma)\pi}{\sin(\alpha+\beta+\gamma')\pi}\,e^{-\pi i(\alpha'-\alpha)} = \frac{\alpha_{\beta'}}{\alpha'_{\beta'}}\,\frac{\sin(\alpha'+\beta'+\gamma)\pi}{\sin(\alpha'+\beta'+\gamma')\pi}\,e^{-\pi i(\alpha'-\alpha)}$$

From 7) and 8) are obtained the two following values for the ratio

$$\frac{\alpha_\beta}{\alpha'_\beta} : \frac{\alpha_{\beta'}}{\alpha'_{\beta'}}$$

$$e^{[illegible]}\,\frac{\sin(\alpha+\beta+\gamma')\pi\,\sin(\alpha'+\beta'+\gamma)\pi}{\sin(\alpha'+\beta'+\gamma')\pi\,\sin(\alpha+\beta+\gamma)\pi},\ [illegible]\ \frac{\sin(\alpha+\beta+\gamma)\pi\,\sin(\alpha'+\beta'+\gamma')\pi}{\sin(\alpha'+\beta'+\gamma)\pi\,\sin(\alpha+\beta+\gamma')\pi}$$

But since [illegible] the first value may be

written [illegible]

and remembering that [illegible]

this is seen to be the same as the second value.

The four relations in 7) and 8) are all included in the symbolic expression

[illegible]

which in fact we actually employed under the form

[illegible]

Having thus three of the ratios

[illegible]

expressed in terms of the fourth, it is apparent that three of the quantities [illegible]

may be expressed in terms of the remaining five. Clearly, there are no more relations between them. for [illegible] applied to each of the equations -

$$
9)\quad \left\{\begin{aligned}
\rho\alpha &= \alpha_\beta\,\rho\beta + \alpha_{\beta'}\,\rho\beta' \\
\rho\alpha' &= \alpha'_\beta\,\rho\beta + \alpha'_{\beta'}\,\rho\beta' \\
\rho\alpha &= \alpha_\gamma\,\rho\gamma + \alpha_{\gamma'}\,\rho\gamma' \\
\rho\alpha' &= \alpha'_\gamma\,\rho\gamma + \alpha'_{\gamma'}\,\rho\gamma'
\end{aligned}\right.
$$

can give one, and only one, relation.

But from the relation [illegible]

where [illegible] and [illegible] are arbitrary, together with the relations 9), when a particular value is assigned to [illegible], they may be completely determined ; and so determined that each shall be finite.

If P_1 is a function with the same exponents as P, then by a proper choice of the initial values and arbitrary constants, we may make any selected five of the quantities

[illegible]

the same in each. The remaining three will then, as already seen, be the same in each, and we shall have the following identical relations:

$$P^{\alpha} = \alpha_{\beta}\, P^{\beta} + \alpha_{\beta'}\, P^{\beta'}, \qquad P^{\alpha'} = \alpha'_{\beta}\, P^{\beta} + \alpha'_{\beta'}\, P^{\beta'}$$

$$P_1^{\alpha} = \alpha_{\beta}\, P_1^{\beta} + \alpha_{\beta'}\, P_1^{\beta'}, \qquad P_1^{\alpha'} = \alpha'_{\beta}\, P_1^{\beta} + \alpha'_{\beta'}\, P_1^{\beta'}$$

whence

$$\begin{vmatrix} P^{\alpha}, & P_1^{\alpha} \\ P^{\alpha'}, & P_1^{\alpha'} \end{vmatrix} = \begin{vmatrix} \alpha_{\beta}, & \alpha_{\beta'} \\ \alpha'_{\beta}, & \alpha'_{\beta'} \end{vmatrix} \begin{vmatrix} P^{\beta}, & P_1^{\beta} \\ P^{\beta'}, & P_1^{\beta'} \end{vmatrix}.$$

identically.

And in precisely the same way,

$$\begin{vmatrix} P^{\alpha}, & P_1^{\alpha} \\ P^{\alpha'}, & P_1^{\alpha'} \end{vmatrix} = \begin{vmatrix} \alpha_{\gamma}, & \alpha_{\gamma'} \\ \alpha'_{\gamma}, & \alpha'_{\gamma'} \end{vmatrix} \begin{vmatrix} P^{\gamma}, & P_1^{\gamma} \\ P^{\gamma'}, & P_1^{\gamma'} \end{vmatrix}.$$

We notice that if [illegible] be multiplied by x it becomes a uniform function in the region of x [illegible] which is neither 0 nor [illegible] for x [illegible] : the same is true for [illegible] for X [illegible] ; and of [illegible]

for X= 1. [illegible]

Writing therefore

10) [illegible]

the first member is clearly uniform, and continuous at the the point 0; the second at the point 1 ; hence both members are uniform and continuous in the region of 0 and 1. But

[illegible] is uniform and continuous in the region of ∞ ; the same is therefore true of [illegible] which, for X $=\infty$, contains the factor [illegible]

Consequently, [illegible] has no singular points. It is therefore a constant.

Recurring to the identical relation

11) [illegible]

[illegible]

[illegible]

of which the second member vanishes, for X [illegible] , since

$(P^\beta$ is finite, for X = . Hence

vanishes, for X = .

Its value is therefore always 0, and the inference is immediate that whence

In the same way we find

the third member being obtained by combining the ratios.

Likewise

Now is uniform and continuous at the point 0 ; at the point ; and at the point 1 ; hence is everywhere uniform and continuous; unless for some value of X other than 0, ∞, or 1, P^α and P both vanish, in which case all these relations become illusory. But this cannot happen, for by aid of equations 9) we may write

$$P^\alpha \frac{dP^{\alpha'}}{dx} - P^{\alpha'} \frac{dP^\alpha}{dx} = M\left(P^\beta \frac{dP^{\beta'}}{dx} - P^{\beta'} \frac{dP^\beta}{dx}\right) = \left(P^\gamma \frac{dP^{\gamma'}}{dx} - P^{\gamma'} \frac{dP^\gamma}{dx}\right)$$

where M = And since

$$P^\alpha P^{\alpha'} X^{-\alpha-\alpha'}$$

does not vanish for X = 0, it follows that P P is zero of

the order $\alpha + \alpha'$ for X = 0, and consequently that

$$P^{\alpha}\frac{dP^{\alpha'}}{dx} - P^{\alpha'}\frac{dP^{\alpha}}{dx}$$

is zero of order $\alpha + \alpha' - 1$ for X = 0.

Likewise, $P^{\beta}\frac{dP^{\beta'}}{dx} - P^{\beta'}\frac{dP^{\beta}}{dx}$ is zero of order $\beta + \beta' +$ for $\frac{1}{X} = 0$.

For $P^{\beta}P^{\beta'}$ contains $\frac{1}{x}$ to the power $\beta + \beta'$ when x is large, and differentiating we introduce the factor $\frac{1}{X}$ once more. Finally, $P^{\gamma}\frac{dP^{\gamma'}}{dx} - P^{\gamma'}\frac{dP^{\gamma}}{dx}$ is zero of order $\gamma + \gamma' - 1$ for X = 1.

Reasoning now precisely as upon equations 10) and 11), we find that

$$\left(P^{\alpha}\frac{dP^{\alpha'}}{dx} - P^{\alpha'}\frac{dP^{\alpha}}{dx}\right)x^{-\alpha-\alpha'+1}(1-x)^{-\gamma-\gamma'+1}$$

is everywhere uniform and continuous, and therefore a constant. If its value were 0, we should have $\frac{1}{P^{\alpha}}\frac{dP^{\alpha}}{dx} = \frac{1}{P^{\alpha'}}\frac{dP^{\alpha'}}{dx}$, that is $\log P^{\alpha} = \log P^{\alpha'} + \text{const}$,

~~(12)~~

But $P^{\alpha} = x^{\alpha}\Gamma^{\alpha}$, $P^{\alpha'} = x^{\alpha'}\Gamma^{\alpha'}$ and equation 12) leads us to the following

$$\alpha \log x - \alpha' \log x + \log \Gamma^{\alpha} - \log \Gamma^{\alpha'} = \text{const}$$

which must hold for any value of $\log x$ whatever. This can only be the case if $\alpha = \alpha'$ which is contrary to hypothesis.

If now, P^{α} and $P^{\alpha'}$ were simultaneously 0 for any value of X other than 0, ∞, 1, the value of the constant

$$\left(P^{\alpha}\frac{dP^{\alpha'}}{dx} - P^{\alpha'}\frac{dP^{\alpha}}{dx}\right) x^{-\alpha-\alpha'+1}(1-x)^{-\gamma-\gamma'+1}$$

would necessarily be zero ; hence, $P^{\alpha'}$ and P^{α} cannot so vanish. We therefore infer that $\frac{P_1^{\alpha}}{P^{\alpha}}$ is a constant.

We are thus led to the theorem :

If two P - functions have the same exponents, the branches of each corresponding to the same exponent can differ only by a constant factor.

We have then -

$$\frac{P_1^{\alpha}}{P^{\alpha}} = \frac{P_1^{\alpha'}}{P^{\alpha'}} = \frac{P_1^{\beta}}{P^{\beta}} = \frac{P_1^{\beta'}}{P^{\beta'}} = \frac{P_1^{\gamma}}{P^{\gamma}} = \frac{P_1^{\gamma'}}{P^{\gamma'}} = g, \quad \text{a constant.}$$

Therefore $\quad P_1^{\alpha} = g P^{\alpha}, \quad P_1^{\alpha'} = g P^{\alpha'}$

and $\quad P_1 = c_{\alpha} g P^{\alpha} + c_{\alpha'} g P^{\alpha'} = c_1 P^{\alpha} + c_2 P^{\alpha'}$

That is, as was previously observed, Riemann's definition determines the P - function to within two arbitrary constants.

Recurring to equations 7) and 8) it may be noticed that the numerators differ from the denominators only by containing γ' instead of α'. Therefore it is evident that to

increase or decrease $\beta, \beta', \gamma, \gamma'$ by any integers whatever, cannot alter the values of the ratios. As to α, α', if one of them be increased or decreased by an odd, and the other by an even integer, the fraction will change sign, but $e^{\pi i(\alpha-\alpha')}$ will change sign at the same time. If both α and α' be increased by odd or even integers, there will be no change of sign in either factor. Hence, to alter the exponents of the P-functions by any integers whatever, will not alter the ratios.

Therefore, if in two P-functions whose exponents differ only by integers, we assume the five arbitrary quantities $a_{\beta}, a_{\beta'}, a'_{\beta}, a'_{\beta'}, a_{\gamma}$ the same in each, as we may, then the remaining three $a_{\gamma'}, a'_{\gamma}, a'_{\gamma'}$ as determined by equations 7) and 8), will be the same in each.

Calling the two functions $P\begin{Bmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{Bmatrix}$ and $P_1\begin{Bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_1' & \beta_1' & \gamma_1' \end{Bmatrix}$ we shall have from the equations -

$$13)\quad \begin{cases} P^{\alpha} = a_{\beta} P^{\beta} + a_{\beta'} P^{\beta'} = a_{\gamma} P^{\gamma} + a_{\gamma'} P^{\gamma'} \\ P^{\alpha'} = a'_{\beta} P^{\beta} + a'_{\beta'} P^{\beta'} = a'_{\gamma} P^{\gamma} + a'_{\gamma'} P^{\gamma'} \\ P_1^{\alpha_1} = a_{\beta} P_1^{\beta_1} + a_{\beta'} P_1^{\beta_1'} = a_{\gamma} P_1^{\gamma_1} + a_{\gamma'} P_1^{\gamma_1'} \\ P_1^{\alpha_1'} = a'_{\beta} P_1^{\beta_1} + a'_{\beta'} P_1^{\beta_1'} = a'_{\gamma} P_1^{\gamma_1} + a'_{\gamma'} P_1^{\gamma_1'}, \end{cases}$$

the resulting equation

$$14)\quad P^{\alpha} P_1^{\alpha_1'} - P^{\alpha'} P_1^{\alpha_1} = c_1 \left(P^{\beta} P_1^{\beta_1'} - P^{\beta'} P_1^{\beta_1}\right) = c_2 \left(P^{\gamma} P_1^{\gamma_1'} - P^{\gamma'} P_1^{\gamma_1}\right)$$

For greater clearness, suppose that $\alpha+\alpha', \beta+\beta', \gamma+\gamma'$, exceed $\alpha_1+\alpha_1', \beta_1+\beta_1', \gamma_1+\gamma_1'$, respectively, by positive

integers. Then by considering the first and third numbers of 14) we conclude that

$$(P^{\alpha}P_1^{\alpha_1'} - P^{\alpha'}P_1^{\alpha_1}) / X^{-\alpha'-\alpha_1}(1-X)^{-\gamma-\gamma_1}$$

is uniform and continuous in the regions of the points 0 and 1; and for all finite values of X; and, by considering the second number, we find that it is infinite of the order $-\alpha'-\alpha_1-\gamma-\gamma_1-\beta'-\beta_1$ for $X = \infty$. It is therefore an entire function of X of degree $-\alpha'-\alpha_1-\gamma-\gamma_1-\beta'-\beta_1$; which number is an integer. Designate this function by F.

Now $\alpha-\alpha'+\beta-\beta'+\gamma-\gamma' = \alpha+\alpha'+\beta+\beta'+\gamma+\gamma'-2(\alpha'+\beta'+\gamma')$

$$= 1-2(\alpha'+\beta'+\gamma'),$$

Hence, when $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are altered by integers, the sum of the differences $\alpha-\alpha'=\lambda$; $\beta-\beta'=\mu$; $\gamma-\gamma'=\nu$ changes by twice the sum of increments of α', β' and γ', that is, by an even number. Put also

$$\alpha_1-\alpha_1'=\lambda_1, \quad \beta_1-\beta_1'=\mu_1, \quad \gamma_1-\gamma_1'=\nu_1$$

and let $\Delta\lambda, \Delta\mu, \Delta\nu$ designate the absolute values of

$$\lambda-\lambda_1, \quad \mu-\mu_1, \quad \nu-\nu_1,$$

and, to fix the ideas, suppose as before, that $\alpha'+\alpha_1'$, $\beta+\beta_1'$, $\gamma+\gamma_1'$ exceed $\alpha_1+\alpha'$, $\beta_1+\beta'$, $\gamma_1+\gamma'$ by positive integers; then

$$\alpha_1+\alpha' = \frac{\alpha+\alpha_1'+\alpha_1+\alpha'}{2} - \frac{\Delta\lambda}{2}$$

Hence we see that

$$-\alpha' - \alpha_1 = -\frac{\alpha + \alpha_1' + \alpha' + \alpha_1}{2} + \frac{J\lambda}{2}$$

$$-\beta' - \beta_1 = -\frac{\beta + \beta_1' + \beta' + \beta_1}{2} + \frac{J\mu}{2}$$

$$-\gamma' - \gamma_1 = -\frac{\gamma + \gamma_1' + \gamma' + \gamma_1}{2} + \frac{J\nu}{2}$$

Hence, the degree of the function F, is $\frac{J\lambda + J\mu + J\nu}{2} - 1.$

Furthermore, if $P\begin{pmatrix}\alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma'\end{pmatrix}$, $P_1\begin{pmatrix}\alpha_1 & \beta_1, & \gamma_1 \\ \alpha_1' & \beta_1', & \gamma_1'\end{pmatrix}$, $P_2\begin{pmatrix}\alpha_2, & \beta_2, & \gamma_2 \\ \alpha_2', & \beta_2', & \gamma_2'\end{pmatrix}$

are three P-functions, whose exponents differ only by integers, we observe, by what preceeds, that in the identical equation,

$$P^{\alpha}(P_1^{\alpha_1'}P_2^{\alpha_2} - P_1^{\alpha_1}P_2^{\alpha_2'}) + P_1^{\alpha_1}(P_2^{\alpha_2}P^{\alpha'} - P_2^{\alpha_2'}P^{\alpha}) + P_2^{\alpha_2}(P^{\alpha'}P_1^{\alpha_1} - P^{\alpha}P_1^{\alpha_1'}) = 0$$

the coefficients of P^{α}, $P_1^{\alpha_1}$, and $P_2^{\alpha_2}$ are entire functions of X.

But P = C,

in the region of the point X = 0 : ~~or as it is more briefly written~~

hence, in the region of the point

which has evidently and for exponents.

In this way, we see that the exponents of P and differ only by integers ; hence

An identical relation in which the coefficients are rational functions of X, exists between any P-function and its first and second differential coefficients. In other words, the P-function satisfies a linear differential equation of the second order.

PART II. The Differential Equation satisfied by the P-function.

Section 1. The properties of the P-function stated as properties of an integral.

1. P is a regular integral of its differential equation. This results from the fact that a being any one of the singular points of P, it has in the region of a the form

$\overline{Y}$ and $\overline{\Gamma}$ being neither zero nor infinite for x and this is by definition the characteristic of a regular integral.

2. The quantities and are the roots of the indicial equations in the regions of the points and respectively. For, these roots are the negative exponents of the factors by which the integrals in the regions of those points must be multiplied to make them uniform, finite,

and continuous, and not zero at the points ; and this is the property of the exponents in the region of in the region of and in the region of .

3. - The points a, b, and c, are critical points of the coefficients of the equations ; because only the critical points of the coefficients, can be branch points of the integrals ; and because the coefficients are rational, these critical points must be poles.

Also, it is possible to determine a differential equation, whose coefficients have no other critical points than a, b, c, of which the P-function is the general integral. Hence, any other equation having coefficients with ~~more~~ critical points, and P for an integral, will not be irreducible.

4. - Since none of the differences is an ~~integral~~, the integral P is free from logarithms.

The familiar properties of the regular linear differential equation here utilized, will be found stated in

Section 2. Theorem.

The indicial equation corresponding to any pole of the coefficients of a linear differential equation of the second order, with regular integrals, is not altered when the posi-

tion of the pole is changed by a linear transformation of the form $x' = \frac{ex+f}{gx+h}$.

Let 1) $\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = 0$ be the equation in question.

Putting $x = \frac{-hx'+f}{gx'-e}$ and making the necessary computations, we find

$$\frac{dy}{dx} = -\frac{(gx'-e)^2}{gf-he} \cdot \frac{dy}{dx'}$$

$$\frac{d^2y}{dx^2} = \frac{(gx'-e)^4}{(gf-he)^2} \cdot \frac{d^2y}{dx'^2} + \frac{2g(gx'-e)^3}{(gf-he)^2} \cdot \frac{dy}{dx'}.$$

Since the integrals are regular, we notice that

$$p_1 = \frac{\varphi(x)}{x-a} = \frac{gx'-e}{ga+h} \cdot \frac{\varphi(x)}{a'-x'} ;$$

where $a' = \frac{f+ae}{h+ag}$;

and

$$p_2 = \frac{\psi(x)}{(x-a)^2} = \frac{(gx'-e)^2}{(ga+h)^2} \cdot \frac{\psi(x)}{(a'-x')^2}$$

Hence, equation 1) becomes

$$2)\quad \frac{d^2y}{dx'^2} + \frac{dy}{dx'}\left[\frac{2g}{gx'-e} - \frac{gf-he}{(gx'-e)(ga+h)} \cdot \frac{\varphi(x)}{a'-x'}\right]$$

$$+ \frac{(gf-he)^2}{(ga+h)^2} \cdot \frac{1}{(gx'-e)^2} \cdot \frac{\psi(x)}{(a'-x')^2} y = 0,$$

observing that $x-a = (a'-x')\frac{h+ag}{gx'-e}$.

~~which may be written~~

2)

~~Since X~~ -

If now $(X-a)\,p_1 = a_0$ and $(X-a)^2 p_2 = c_0$, when $X = a$,

the indicial equation for the point a is

$$r(r-1) + a_0 r + c_0 = 0 = r(r-1) + r\,\varphi(a) + \psi(a)$$

Calling q_1 and q_2 the coefficients of 2), we must find the values of $(x'-a')q_1$ and $(x'-a')^2 q_2$ when $x' = a'$; since a' is the new pole, and x' = a' when x = a.

We have

$$(x'-a')q_1 = \frac{gf-he}{ga'-e}\,\varphi(a)\,\frac{1}{ga+h}$$

$$(x'-a')^2 q_2 = \frac{(gf-he)^2}{(ga'-e)^2}\,\frac{\psi(a)}{(ga+h)^2}$$

when $x' = a'$

But

$$ga'-e = \frac{gf-he}{ga+h}. \text{ Hence}$$

$$(x'-a')q_1 = \varphi(a)$$

$$(x'-a')^2 q_2 = \psi(a)$$

when $x' = a'$. Hence the new indicial equation is the same as the old one.

Section 3. - The transformation $x = \dfrac{-hx'+f}{gx'-e}$, or $x' = \dfrac{ex+f}{gx+h}$.

When $x = a$, $x' = \dfrac{ea+f}{ga+h} = a'$

$x = b$, $x' = \dfrac{eb+f}{gb+h} = b'$

$x = c$, $x' = \dfrac{ec+f}{gc+h} = c'$

$$\frac{1}{x-a} = \frac{gx'-e}{ga+h} \cdot \frac{1}{a'-x'}$$

$$\frac{1}{x-b} = \frac{gx'-e}{gb+h} \cdot \frac{1}{b'-x'}$$

$$\frac{1}{x-c} = \frac{gx'-e}{gc+h} \cdot \frac{1}{c'-x'}$$

Hence, the poles are merely changed in position by this transformation, and by assigning suitable values to the constants, we may place them at pleasure in the plane.

1. If, $f = -ac$
$h = -gb$
$e(c-a) = g(c-b)$, or $g = \dfrac{e(c-a)}{c-b}$; ~~then~~

then $a' = 0$, $b' = \infty$, $c' = 1$, and
$$x' = \frac{e(x-a)}{g(x-b)}.$$

2. Taking $c = 1$ these results are simplified :

$$f = -a$$

$$g = \frac{c-a}{c-b}$$

$$h = -b\,\frac{c-a}{c-b}$$

$$a' = 0,\ b' = \infty,\ c' = 1$$

$$\lambda' = \frac{c-b}{c-a}\cdot\frac{x-a}{x-b}.$$

Thus, by this transformation, without changing the indicial equations, and therefore without changong the exponents $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ we may place the branch points of the integral P at $0, \infty$ and 1. Any one of the branch points 0, ∞ and 1, may be made to correspond to any pair of exponents $\alpha, \alpha';\ \beta, \beta';\ \gamma, \gamma'$.

For the possible arrangements are

1) $0,\ \infty,\ 1$
2) $0,\ 1,\ \infty$
3) $1,\ \infty,\ 0$
4) $1,\ 0,\ \infty$
5) $\infty,\ 0,\ 1$
6) $\infty,\ 1,\ 0.$

For 2) we must take $c' = \infty$, $f = -a$, $b - a = gb + h$, $gc + h = 0$

or $h = -cg$; $b - a = g(b - c)$; $g = \dfrac{b-a}{b-c}$; $h = -c\,\dfrac{b-a}{b-c}$.

$$\lambda' = \frac{\lambda - a}{\lambda - c}\cdot\frac{b-c}{b-a}.$$

For 3)

$$a + f' = ga + h'$$
$$gb + h' = 0$$
$$c + f' = 0$$
$$a - c = g(a - b), \text{ or } g = \frac{a-c}{a-b}; \; f' = -c, \; h' = -b\,\frac{a-c}{a-b}.$$
$$\lambda''' = \frac{\lambda - c}{\lambda - b} \cdot \frac{a-b}{a-c}.$$

For 4)

$$a + f' = ga + h'$$
$$b + f' = 0$$
$$gc + h' = 0$$
$$a - b = g(a - c); \; g = \frac{a-b}{a-c}; \; f' = -b; \; h' = -c\,\frac{a-b}{a-c}.$$
$$\lambda^{II} = \frac{\lambda - b}{\lambda - c} \cdot \frac{a-c}{a-b}.$$

For 5)

$$ga + h = 0$$
$$b + f' = 0$$
$$c + f' = gc + h'$$
$$c - b = g(c - a); \; g = \frac{c-b}{c-a}; \; f' = -b; \; h' = -a\,\frac{c-b}{c-a}.$$
$$\lambda^{I} = \frac{\lambda - b}{\lambda - a} \cdot \frac{c-a}{c-b}.$$

For 6)

$$ga + h' = 0, \quad b + f' = gb + h', \quad c + f' = 0.$$
$$b - c = g(b - a); \; g = \frac{b-c}{b-a}; \; f' = -c; \; h' = -a\,\frac{b-c}{b-a}.$$
$$\lambda^{II} = \frac{\lambda - c}{\lambda - a} \cdot \frac{b-a}{b-c}.$$

These variables satisfy the following equations;

$$\lambda^{I} = \frac{1}{\lambda'}$$

$$\lambda^{II} = \frac{1}{\lambda'}$$

$$\lambda^{III} = \frac{1}{\lambda^{II}}$$

$$\lambda^{IV}\lambda^{V} = -\lambda'$$

$$\lambda^{V} / = -\lambda'$$

Whence ;

$$\lambda^{II} = \frac{\lambda' - 1}{\lambda'}$$

$$\lambda^{I} = \frac{1}{\lambda'}$$

$$\lambda^{II} = \frac{1}{1-\lambda'}$$

$$\lambda^{IV} = -\lambda'$$

$$\lambda^{V} = -\frac{\lambda'}{1-\lambda'}$$

Thus all of them may be unambiguously expressed in terms of λ' and we may obtain the six forms of P mentioned in section 3, Part1.

Section 4. - Effect upon the indicial equation when the integral is multiplied by a factor of the form $(x-a)^{\rho_1}(x-b)^{\rho_2}(x-c)^{\rho_3}$

Let the differential equation be

$$1) \quad \frac{d^2y}{dx^2} + \frac{\varphi(x)}{x-a}\frac{dy}{dx} + \frac{\psi(x)}{(x-a)^2}y = 0 = \frac{d^2y}{dx^2} + p\frac{dy}{dx} + q\,y,$$

where only the pole $x = a$ is brought into evidence, and let y be an integral in the region of the point a. The corres-

sponding indicial equation is

$$r(r-1) + r\varphi(a) + \psi(a) = 0, \text{ or}$$
$$r^2 + r[\varphi(a)-1] + \psi(a) = 0.$$

If α and α' are the roots, then

$$\alpha + \alpha' = 1 - \varphi(a)$$
$$\alpha\alpha' = \psi(a).$$

If now the equation satisfied by

$$(x-a)^{\delta_1}(x-b)^{\delta_2}(x-c)^{\delta_3}\, y = z \qquad \text{is}$$

$$2) \qquad \frac{d^2z}{dx^2} + p\frac{dz}{dx} + qz = 0$$

and if $(x-a)^{\delta_1}(x-b)^{\delta_2}(x-c)^{\delta_3} = \varphi$, then

$$\frac{d^2z}{dx^2} = \varphi\frac{d^2y}{dx^2} + 2\varphi\frac{dy}{dx}\left(\frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c}\right)$$
$$- \varphi y\left(\frac{\delta_1}{(x-a)^2} + \frac{\delta_2}{(x-b)^2} + \frac{\delta_3}{(x-c)^2}\right) + \varphi y\left(\frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c}\right)^2,$$

and $$\frac{dz}{dx} = \varphi\frac{dy}{dx} + \varphi y\left(\frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c}\right).$$

Thus equation (2) becomes after dividing out the factor φ,

$$3) \qquad \frac{d^2y}{dx^2} + \frac{dy}{dx}\left[p + 2\left(\frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c}\right)\right]$$
$$+ y\left[\left(\frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c}\right)^2 - \frac{\delta_1}{(x-a)^2} - \frac{\delta_2}{(x-b)^2} - \frac{\delta_3}{(x-c)^2} + p\left(\frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c}\right) + q\right] = 0$$

This can be no other than the equation

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + q_1 y = 0$$

and therefore, identifying the coefficients,

$$p_1 = q_1 + 2\left(\frac{\delta_1}{x-a}+\frac{\delta_2}{x-b}+\frac{\delta_3}{x-c}\right)$$

$$p_2 = \left(\frac{\delta_1}{x-a}+\frac{\delta_2}{x-b}+\frac{\delta_3}{x-c}\right)^2 - \frac{\delta_1}{(x-a)^2} - \frac{\delta_2}{(x-b)^2} - \frac{\delta_3}{(x-c)^2} + q_1\left(\frac{\delta_1}{x-a}+\frac{\delta_2}{x-b}+\frac{\delta_3}{x-c}\right) + q_2$$

That is

$$q_1 = p_1 - 2\left(\frac{\delta_1}{x-a}+\frac{\delta_2}{x-b}+\frac{\delta_3}{x-c}\right)$$

$$q_2 = p_2 - \left(\frac{\delta_1}{x-a}+\frac{\delta_2}{x-b}+\frac{\delta_3}{x-c}\right)^2 + \frac{\delta_1}{(x-a)^2}+\frac{\delta_2}{(x-b)^2}+\frac{\delta_3}{(x-c)^2} - q_1\left(\frac{\delta_1}{x-a}+\frac{\delta_2}{x-b}+\frac{\delta_3}{x-c}\right)$$

Hence, a, is a pole of the same degree of multiplicity for the new coefficients, as for the old. To form the indicial equation, we calculate $(x-a)q_1$ and $(x-a)^2 q_2$ for $x = a$

$$(x-a)q_1 = \varphi(a) - 2\delta_1$$

$$(x-a)^2 q_2 = \psi(a) - \delta_1^2 + \delta_1 - \delta_1[\varphi(a) - 2\delta_1]$$

and the equation is

$$r^2 + r[\varphi(a) - 2\delta_1 - 1] + \psi(a) - \delta_1\varphi(a) + \delta_1^2 + \delta_1 = 0$$

If the roots of this be σ and σ', then

$$\sigma + \sigma' = 1 - 2\delta_1 - \varphi(a)$$

$$\sigma\sigma' = \psi(a) - \delta_1\varphi(a) + \delta_1 + \delta_1^2$$

Hence,

$$\sigma + \sigma' = \alpha + \alpha' + 2\delta_1 = (\alpha + \delta_1) + (\alpha' + \delta_1)$$

$$\sigma\sigma' = \alpha\alpha' + \delta_1(\alpha + \alpha') + \delta_1^2 = (\alpha + \delta_1)(\alpha' + \delta_1)$$

Whence, we conclude that

$$\sigma = \alpha - \delta_1$$
$$\sigma = \alpha + \delta_1$$

Therefore, to multiply the integral of equation 1) by the factor

$$(x-a)^{\delta_1}(x-b)^{\delta_2}(x-c)^{\delta_3}$$

increases the roots of the indicial equation corresponding to each pole, by the exponent of the corresponding factor ; but leaves their difference unchanged.

Suppose the point infinity is a singular point of the integral of equation 1:- the other singular points being a and b.

If we multiply v , the integral, by $(x-a)^{\delta_1}(x-b)^{\delta_2}$, the roots of the indicial equations at a and b will be increased by δ_1 and δ_2 respectively, as we have seen, and the transformed equation 3) becomes

$$4)\quad \frac{d^2u}{dx^2} + \frac{du}{dx}\left[p_1 - 2\left(\frac{\delta_1}{x-a} + \frac{\delta_2}{x-b}\right)\right] + u\left[\left(\frac{\delta_1}{x-a} + \frac{\delta_2}{x-b}\right)^2 + \frac{\delta_1}{(x-a)^2} + \frac{\delta_2}{(x-b)^2} - p_1\left(\frac{\delta_1}{x-a} + \frac{\delta_2}{x-b}\right) + p_2\right] = 0.$$

In equation 1) let us make the transformation

$$x = \frac{1}{x'} \quad \text{or} \quad x' = \frac{1}{x}$$

Whence

$$\frac{du}{dx} = -x'^2 \frac{du}{dx'}$$
$$\frac{d^2u}{dx^2} = 2x'^3\frac{du}{dx'} + x'^4\frac{d^2u}{dx'^2}$$

and equation 1) becomes

5) $$\frac{d^2 y'}{dx'^2} + \frac{dy'}{dx'}\left[\frac{2}{x'} - \frac{p_1'}{x'^2}\right] + \frac{p_2'}{x'^4} y' = 0$$

Of this equation, X' = 0, must be a pole, by hypothesis, and the corresponding indicial equation is, if

$$\frac{p_1'}{x'^2} = \frac{\varphi(x')}{x'} \; ; \; \frac{p_2'}{x'^4} = \frac{\psi(x')}{x'^2}$$

6) $$r^2 + r\left[\varphi(0) + 1\right] + \psi(0) = 0$$

Again

$$q_1' = p_1' - 2\frac{\delta_1 x'}{1-ax'} - 2\frac{\delta_2 x'}{1-bx'}$$

$$q_2' = p_2' - \left[\left(\frac{\delta_1}{1-ax} + \frac{\delta_2}{1-bx'}\right)^2 - \frac{\delta_1}{(1-ax')^2} - \frac{\delta_2}{(1-bx')^2}\right]x'^2 - p_1' x'\left(\frac{\delta_1}{1-ax'} + \frac{\delta_2}{1-bx'}\right)$$

Transforming 2) by the substitution $x = \frac{1}{x'}$ it becomes

7) $$\frac{d^2 z'}{dx'^2} + \frac{dz'}{dx'}\left[\frac{2}{x'} - \frac{q_1'}{x'^2}\right] + \frac{q_2'}{x'^4} z' = 0 .$$

and showing that, for $x' = 0$,

$$\frac{q_1'}{x'^2} \cdot x' = \varphi(0) - 2\delta_1 - 2\delta_2$$

$$\frac{q_2'}{x'^4} \cdot x'^2 = \psi(0) + (\delta_1 + \delta_2)^2 + (\delta_1 + \delta_2)(1 - \varphi(0))$$

~~and~~ The indicial equation corresponding to the point $x = 0$ is

8) $r^2 + r[-\varphi(0) + 2\delta_1 + 2\delta_2 + 1] + \psi(0) + (\delta_1 + \delta_2)^2 + (\delta_1 + \delta_2)[1 - \varphi(0)] = 0.$

Calling the roots of equation 8) σ and σ', we find

$$\sigma + \sigma' = \varphi(0) - 1 - 2\delta_1 - 2\delta_2$$

$$\sigma\sigma' = \psi(0) + (\delta_1 + \delta_2)(1 - \varphi(0)) + (\delta_1 + \delta_2)^2$$

Again, if the roots of equation 6) are α and α'

$$\alpha + \alpha' = \varphi(0) - 1$$

$$\alpha\alpha' = \psi(0).$$

Hence,

$$\sigma + \sigma' = (\alpha - \delta_1 - \delta_2) + (\alpha' - \delta_1 - \delta_2)$$

$$\sigma\sigma' = (\alpha - \delta_1 - \delta_2)(\alpha' - \delta_1 - \delta_2).$$

Whence

$$\sigma = \alpha - \delta_1 - \delta_2$$

$$\sigma' = \alpha' - \delta_1 - \delta_2$$

Therefore, when the singular points of the integral, are a, b, ∞, to multiply the integral by the factor $(x - a)^{\delta_1}(x - b)^{\delta_2}$ diminishes the roots of the indicial equation for the point ∞ by $\delta_1 + \delta_2$ and leaves their difference unchanged.

In order that neither coefficient become infinite for $x' = 0$ we must have

$$\lambda - B = 0$$
$$C_1 = D_1 = 0$$

Hence ~~the~~ equation 1) reduces to

$$3)\quad \frac{d^2u}{dx^2} + \frac{P + \alpha x + \lambda x^2}{(x-a)(x-b)(x-c)}\frac{du}{dx} + \frac{P_1 + A_1 x + B_1 x^2}{(x-a)^2(x-b)^2(x-c)^2}\,u = 0$$

Which may be written

$$4)\quad \frac{d^2u}{dx^2} + \frac{du}{dx}\left[\frac{\mathfrak{A}}{x-a} + \frac{\mathfrak{B}}{x-b} + \frac{\mathfrak{C}}{x-c}\right] + \frac{u}{(x-a)(x-b)(x-c)}\left[\frac{\mathfrak{A}_1}{x-a} + \frac{\mathfrak{B}_1}{x-b} + \frac{\mathfrak{C}_1}{x-c}\right] = 0$$

where $\mathfrak{A}, \dots, \mathfrak{A}_1, \dots$ are new constants.

The indicial equations at the points, a, b, c, are respectively,

$$r^2 + r(\mathfrak{A} - 1) + \frac{\mathfrak{A}_1}{(a-b)(a-c)} = 0$$
$$r^2 + r(\mathfrak{B} - 1) + \frac{\mathfrak{B}_1}{(b-a)(b-c)} = 0$$
$$r^2 + r(\mathfrak{C} - 1) + \frac{\mathfrak{C}_1}{(c-a)(c-b)} = 0,$$

and by hypothesis, their roots are $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ respectively. Therefore,

$$\mathfrak{A} = 1 - \alpha - \alpha';\quad \mathfrak{A}_1 = \alpha\alpha'(a-b)(a-c);$$
$$\mathfrak{B} = 1 - \beta - \beta';\quad \mathfrak{B}_1 = \beta\beta'(b-a)(b-c);$$
~~Similarly~~
$$\mathfrak{C} = 1 - \gamma - \gamma';\quad \mathfrak{C}_1 = \gamma\gamma'(c-a)(c-b).$$

Section 5. Determination of the coefficients of the differential equation satisfied by $P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$.

From the general theory of linear differential equations, since the integral P is regular and has three singular points, the coefficients of the equation must conform to the following conditions :

1. The coefficient of $\frac{dy}{dx}$ will be a rational fraction whose numerator cannot be of a degree exceeding 3 - 1, three being the number of poles; and the denominator is $(x-a)(x-b)(x-c)$.

2. The numerator of the coefficient of y cannot be of higher degree than 2(3-1) = 4; and the denominator is $(x-a)^2(x-b)^2(x-c)^2$.

3. The constants of the coefficients must be so related that after effecting the transformation $x = \frac{1}{x'}$, the point $x' = 0$ shall not be a singular point for the equation; otherwise $x = \infty$ would be a singular point for the original equation, contrary to hypothesis.

In conformity with these conditions, we may assume the equation to be ; $F, A, \ldots; F_1, A_1, \ldots$ being constants,

$$1)\quad \frac{d^2y}{dx^2} + \frac{F + Ax + Bx^2}{(x-a)(x-b)(x-c)}\,\frac{dy}{dx} + \frac{F_1 + A_1x + B_1x^2 + C_1x^3 + D_1x^4}{(x-a)^2(x-b)^2(x-c)^2}\,y = 0.$$

Making the transformation $x = \frac{1}{x'}$, this becomes

$$2)\quad \frac{d^2y}{dx'^2} + \frac{dy}{dx'}\left[\frac{2}{x'} - \frac{1}{x'^2}\cdot\frac{x'^3\left(F + \frac{A}{x'} + \frac{B}{x'^2}\right)}{(1-ax')(1-bx')(1-cx')}\right] + \frac{y}{x'^4}\left[\frac{x'^4\left(F_1 + \frac{A_1}{x'} + \frac{B_1}{x'^2} + \frac{C_1}{x'^3} + \frac{D_1}{x'^4}\right)}{(1-ax')^2(1-bx')^2(1-cx')^2}\right] = 0$$

If $a = 1$, $c = 0$ these values become

$$\alpha'' = 1-\alpha-\alpha'; \quad \alpha_1 = \alpha\alpha'(1-b)$$
$$\beta'' = 1-\beta-\beta'; \quad \beta_1 = \beta\beta'(b-1)b$$
$$\gamma'' = 1-\gamma-\gamma'; \quad \gamma_1 = \gamma\gamma' b$$

and equation 4) takes the form

5)
$$\frac{d^2u}{dx^2} + \frac{du}{dx}\left[\frac{\alpha''}{x-1} + \frac{\beta''}{x-b} + \frac{\gamma''}{x}\right] + \frac{u}{x(x-1)(x-b)}\left[\frac{\alpha_1}{x-1} + \frac{\beta_1}{x-b} + \frac{\gamma_1}{x}\right] = 0.$$

Let us now transform 5) by the substitution
$$x' = \frac{x(1-b)}{x-b}.$$

Such that when
$$x = 1,\ x' = 1$$
$$x = 0,\ x' = 0$$
$$x = b,\ x' = \infty$$

From this
$$x = \frac{bx'}{x'-(1-b)}; \quad x-b = \frac{b(1-b)}{x'-(1-b)}; \quad x-1 = \frac{(b-1)(x'-1)}{x'-(1-b)}.$$
$$\frac{du}{dx} = -\frac{du}{dx'}\cdot\frac{(x'-(1-b))^2}{b(1-b)}.$$
$$\frac{d^2u}{dx^2} = \frac{d^2u}{dx'^2}\,\frac{(x'-(1-b))^4}{b^2(1-b)^2} + 2\frac{du}{dx'}\,\frac{(x'-(1-b))^3}{b^2(1-b)^2}$$

Introducing these values, equation 5) becomes
$$\frac{[x'-(1-b)]^4}{b^2(1-b)^2}\frac{d^2u}{dx'^2} + \frac{du}{dx'}\left[2\frac{[x'-(1-b)]^3}{b^2(1-b)^2} - \frac{[x'-(1-b)]^2}{b(1-b)}\left(\alpha''\frac{x'-(1-b)}{(b-1)(x'-1)} + \beta''\frac{x'-(1-b)}{b(1-b)} + \gamma''\frac{x'-(1-b)}{bx'}\right)\right]$$
$$-\,u\,\frac{[x'-(1-b)]^3}{b^2(1-b)^2x'(x'-1)}\left[\alpha_1\frac{x'-(1-b)}{(b-1)(x'-1)} + \beta_1\frac{x'-(1-b)}{b(1-b)} + \gamma_1\frac{x'-(1-b)}{bx'}\right] = 0.$$

Noting that $\mathcal{L}\frac{x'-(1-b)}{(b-1)(x'-1)} + \mathcal{M}\frac{x'-(1-b)}{b(1-b)} + \mathcal{N}\frac{x'-(1-b)}{bx'} =$

$$\frac{x'-(1-b)}{b(1-b)(x'-1)x'}\left[-\mathcal{L}bx' + \mathcal{M}x'(x'-1) + \mathcal{N}(1-b)(x'-1)\right]$$

$$= \frac{x'-(1-b)}{b(1-b)x'(x'-1)}\left[\mathcal{M}x'^2 + x'\left((1-b)(\mathcal{N}+\mathcal{L}) - \mathcal{L} - \mathcal{M}\right) + \mathcal{N}(1-b)\right]$$

And subtracting the quantity in the bracket from $2x'(x'-1)$ (which has now the same coefficient) we get

6) $(2-\mathcal{M})x'^2 + x'\left[\mathcal{L}+\mathcal{M}-(1-b)(\mathcal{N}+\mathcal{L}) - 2\right] + \mathcal{N}(1-b)$

Now $\mathcal{L}+\mathcal{M}-2 = -\mathcal{N}$

and $\mathcal{N}+\mathcal{L} = 2-\mathcal{M}$,

remembering that $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$.

Hence the expression 6) breaks up into the factors

$$\left[x'-(1-b)\right]\left[(2-\mathcal{M})x'-\mathcal{N}\right].$$

Again, noting that

$$\frac{\mathcal{L}}{b-1} = -\alpha\alpha'; \quad \frac{\mathcal{M}}{b(1-b)} = -\beta\beta'; \quad \frac{\mathcal{N}}{b} = \gamma\gamma'$$

And observing that the factor $\frac{[x'-(1-b)]^2}{b^2(1-b)^2}$ is now common to every term of the equation, we obtain

7) $$\frac{d^2y}{dx'^2} + \frac{dy}{dx'}\,\frac{x'(2-\mathcal{M})-\mathcal{N}}{x'(x'-1)} - \frac{y}{x'(x'-1)}\left[\frac{-\alpha\alpha'}{x'-1} - \beta\beta' + \frac{\gamma\gamma'}{x'}\right]=0,$$

or, finally,

8) $$\frac{d^2y}{dx'^2} + \frac{dy}{dx'}\,\frac{x'(1+(\beta+\beta')) - 1+\gamma+\gamma'}{x'(x'-1)} - \frac{y}{x'(x'-1)}\left[\frac{\alpha\alpha'}{1-x'} - \beta\beta' + \frac{\gamma\gamma'}{x'}\right]=0.$$

If we had chosen $a = 0,\ b = \infty,\ c = 1$, the equation would have been

9) $$\frac{d^2u}{dx^2} + \frac{du}{dx}\,\frac{\alpha+\alpha'-1+x(\gamma+\beta+\beta')}{x(x-1)} + u\,\frac{\alpha\alpha'+x(\gamma\gamma'-\alpha\alpha'-\beta\beta')+\beta\beta' x^2}{x^2(1-x)^2} = 0.$$

which is Papperitz's form.

~~which is Papperitz's form.~~

Section 3. $\alpha = 0,\ \alpha' = \frac{1}{2};\ \beta+\beta'+\gamma+\gamma' = \frac{1}{2}.$

To this case we may reduce that of one difference $= \frac{1}{2}$. Substituting these values in the differential equation, it becomes :

1) $$\frac{d^2u}{dx^2} + \frac{du}{dx}\,\frac{x(1+\beta+\beta')-\frac{1}{2}}{x(x-1)} + \frac{u}{4}\,\frac{\beta\beta' x^2+x(\gamma\gamma'-\beta\beta')}{x^2(1-x)^2} = 0$$

From this, if $x = z^2$, we obtain the equation

2) $$\frac{d^2u}{dz^2} + \frac{du}{dz}\,\frac{z(1+2\beta+2\beta')}{z(z^2-1)} + 4u\,\frac{\beta\beta' z^2+\gamma\gamma'-\beta\beta'}{(1-z^2)^2} = 0$$

~~On~~

~~That is~~

~~41~~

Effecting the transformation $z = \frac{1}{z'}$ in 2) we obtain

3) $$\frac{d^2u}{dz'^2} + \frac{du}{dz'}\left[\frac{2}{z'} - \frac{1+2\beta+2\beta'}{z'(1-z'^2)}\right] + 4u\,\frac{\beta\beta' z'^2 + (\gamma\gamma' - \beta\beta')z'^4}{z'^4(z'^2-1)^2} = 0$$

which shows that $z' = 0$ is a pole of multiplicity 1 for the first, and 2 for the second coefficient. The poles of 2) are consequently $+1, -1,$ and ∞ and the corresponding indicial equations are respectively

4) $$r^2 + r(\beta+\beta'-\tfrac{1}{2}) + \gamma\gamma' = 0$$

5) $$r^2 + r(\beta+\beta'-\tfrac{1}{2}) + \gamma\gamma' = 0$$

6) $$r^2 + r(-2\beta-2\beta') + 4\beta\beta' = 0$$

But, remembering that

$$\beta+\beta'+\gamma+\gamma' = \tfrac{1}{2}$$

these equations reduce to

7) $$r^2 - r(\gamma+\gamma') + \gamma\gamma' = 0$$

8) $$r^2 - r(\gamma+\gamma') + \gamma\gamma' = 0$$

9) $$r^2 - 2r(\beta+\beta') + 4\beta\beta' = 0.$$

of which the roots are respectively

$$r, r'; \; r, r'; \; 2\beta, 2\beta'.$$

That is, by the transformation $x = z^2$ the equation satisfied by $P\begin{pmatrix} 0 & \beta & \gamma & \\ \frac{1}{2} & \beta' & \gamma' & x \end{pmatrix}$ becomes an equation satisfied by

$P\begin{pmatrix} \gamma & 2\beta & \gamma & \\ \gamma' & 2\beta' & \gamma' & z \end{pmatrix}$, which is a result obtained in

PART 1. Section 5.

In equation 10) Section 5., let us make $\alpha' = \beta' = \gamma' = 0$ and $\alpha = \beta = \nu$, while $\gamma = 1$; we thus obtain

10) $$\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1-\nu'}{x} = 0$$

The general integral of this equation is

$$cx^{\nu'} + c'$$

where c and c' are constants. Hence, we conclude with Riemann that $P\begin{pmatrix} 0 & 0 & 0 & \\ \nu' & -\nu' & 1 & x \end{pmatrix} = cx^{\nu'} + c'$.

Section 7. Spherical Harmonics expressed as P-functions.

In equation 2) Section 6, the quantities $\beta, \beta', \gamma, \gamma'$ are connected by the relation $\cancel{\beta + \beta' + \gamma + \gamma' = \frac{1}{2}}$

1) $$\beta + \beta' + \gamma + \gamma' = 1$$

If we make $\gamma' = 0$, 1) becomes $\beta + \beta' + \gamma = 1$

a relation which is identically satisfied by the values

$$\gamma = 0, \ \beta = \frac{n+1}{2}, \ \beta' = -\frac{n}{2}.$$

These values reduce Equation 2) of Section 6. to

$$\frac{d^2u}{dz^2} + \frac{du}{dz}\frac{2z}{z^2-1} - \frac{n(n+1)}{z^2-1}y = 0$$

which is the differential equation satisfied by the zonal spherical harmonics, if we apply that name to the function

$\frac{d^n}{dz^n}(z^2-1)^n$, Jordan, Cours d'Analyse, Vol 1, p. 51.

Hence the P-function $P\begin{Bmatrix} 0 & \infty & -1 & \\ 0 & n+1 & 0 & z \\ 0 & -n & 0 & \end{Bmatrix}$

represents the zonal harmonic of order n. (See Ferrers, Spherical Harmonics, p. 12.)

Section 8. The Toroidal functions expressed as P-functions. In equation 2) of Sec. 6, let us make

$$\gamma + \gamma' = 0$$
$$4\gamma\gamma' = -m^2$$

Then shall $\gamma = -\gamma' = \pm \frac{m}{2}.$

And from the relation $\beta + \beta' + \gamma + \gamma' = \frac{1}{2}$

we find $\beta + \beta' = \frac{1}{2}$

Assuming also $\beta - \beta' = n$

we get $4\beta\beta' = -n^2 + \frac{1}{4}$

and also, $\cdot + 2\beta + 2\beta' = 2$

Substituting these values in equation 2) it becomes,

$$1)\quad \frac{d^2y}{dz^2} + \frac{2}{z^2-1}\frac{du}{dz} + \frac{(n^2-\frac{1}{4})(1-z^2)-m^2}{(1-z^2)^2}y = 0.$$

which is the differential equation satisfied by the Toroidal functions. (Basset, Hydrodynamics, Vol. 2 p.22.)

Observing that $\beta = \frac{n}{2} + \frac{1}{4}$, $\beta' = \frac{1}{4} - \frac{n}{2}$, we see that the P-function

$$P\begin{Bmatrix} 1 & \infty & -1 & \\ \frac{m}{2} & n+\frac{1}{2} & \frac{m}{2} & z \\ -\frac{m}{2} & \frac{1}{2}-n & -\frac{m}{2} & \end{Bmatrix}$$

represents the Toroidal Functions.

In equation 1) making $m = 0$, which corresponds to $\gamma = \gamma' = 0$, we get the equation for zonal toroidal functions,

$$2) \quad \frac{d^2u}{dz^2} + \frac{2}{z^2-1}\frac{du}{dz} + \frac{n^2-\frac{1}{4}}{1-z^2}\,u = 0$$

which has for its integral the P-function

$$P\begin{Bmatrix} 1 & \infty & -1 & \\ 0 & n+\frac{1}{2} & 0 & z \\ 0 & \frac{1}{2}-n & 0 & \end{Bmatrix}$$

Section 9. Bessel's Equation.

In Eq. 10) of Section 8, let us make the substitution

(1) $$x = \varepsilon z$$

then shall
$$\frac{du}{dx} = \frac{1}{\varepsilon}\frac{du}{dz}$$
$$\frac{d^2u}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2u}{dz^2}$$

And the equation becomes

$$(2) \quad \frac{d^2u}{dz^2} + \frac{du}{dz}\cdot\frac{\varepsilon z(1+\beta+\beta') + \alpha+\alpha'-1}{z(\varepsilon z-1)} + u\,\frac{\alpha\alpha' + \varepsilon z(\gamma+\gamma'-\alpha\alpha'-\beta\beta') + \beta\beta'\varepsilon^2 z^2}{z^2(\varepsilon z-1)^2} = 0$$

Let now, ε tend toward zero, and β, β' toward infinity in such a manner that the product $\varepsilon^2 \beta\beta'$ shall be constantly equal to 1. That is, $\beta\beta' = \frac{1}{\varepsilon^2}$

and also let $\alpha\alpha' = -m^2$, a constant

$$\gamma\gamma' = \beta\beta'$$
$$\alpha + \alpha' - 1 = -1$$
$$\beta + \beta' = 0$$

These equations together with

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

enable us to determine the six exponents; thus we find

$$\alpha = \pm m, \quad \alpha' = \mp m$$
$$\beta = \pm \frac{1}{\varepsilon}\sqrt{-1}; \quad \beta' = \mp \frac{1}{\varepsilon}\sqrt{-1}$$
$$\gamma = \tfrac{1}{2}\left(1 \pm \tfrac{1}{\varepsilon}\sqrt{\varepsilon^2-4}\right); \quad \gamma' = \tfrac{1}{2}\left(1 \mp \tfrac{1}{\varepsilon}\sqrt{\varepsilon^2-4}\right).$$

and Eq. 2) becomes

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + y\left(1 - \frac{m^2}{z^2}\right) = 0$$

which is Bessel's equation.

We conclude therefore, that the limiting value of

$$P\begin{Bmatrix} 0 & \infty & 1 \\ \pm m & \pm\frac{1}{\varepsilon}\sqrt{-1} & \frac{1}{2}\left(1 \pm \frac{1}{\varepsilon}\sqrt{\varepsilon^2-4}\right) \\ \mp m & \mp\frac{1}{\varepsilon}\sqrt{-1} & \frac{1}{2}\left(1 \mp \frac{1}{\varepsilon}\sqrt{\varepsilon^2-4}\right) \end{Bmatrix}$$

when $\varepsilon = 0$ is the Bessel's function $J_m(z)$.

Section 10. The P-function expressed as a hypergeo-

metric series.

Eq. 10) of section 6) is

$$(1)\quad \frac{d^2y}{dx^2} + \frac{dy}{dx}\,\frac{x(1+\beta+\beta')+\alpha+\alpha'-1}{x(x-1)} + y\,\frac{\alpha\alpha' + x(\gamma\gamma'-\alpha\alpha'-\beta\beta')+\beta\beta' x^2}{x^2(1-x)^2} = 0.$$

In the region of the point X - o the indicial equation is 2) $r^2 - r(\alpha+\alpha') + \alpha\alpha' = 0$ ~~$r^2 - r(1-\alpha-\alpha'-1) - \alpha\alpha' = 0$~~

of which the roots are α and α'. Knowing that the integral does not contain logarithms, we may assert, that- in the region of the point o, Eq.1) will be satisfied by a convergent series of the form $X^{\alpha}(C_0 + C_1 x + C_2 x^2 + \text{-----} + c_n X^n + \text{---})$

Before substituting this series in Eq.1) it will be convenient to transform the equation as follows - Put $\alpha - \alpha' = \lambda$

$\beta - \beta' = \mu$, $\gamma - \gamma' = \nu$, and, keeping λ, μ, ν constant, let us determine α', β', γ' so that the following conditions may be satisfied :

$$\alpha\alpha' = 0, \quad \gamma\gamma' = 0.$$

Consistently with these conditions we make

$$\alpha' = 0, \ \alpha = \lambda$$
$$\gamma' = 0, \ \gamma = \nu.$$

and accordingly

$$\beta + \beta' = 1 - \lambda - \nu$$
$$\beta - \beta' = \mu$$

~~13.~~

Hence

$$2\beta = 1 - \lambda - \nu + \mu$$
$$2\beta' = 1 - \lambda - \nu - \mu$$
$$\beta\beta' = \frac{(1-\lambda-\nu)^2 - \mu^2}{4}$$

With these values, Eq. 1) becomes

$$4)\qquad \frac{d^2y}{dx^2} + \frac{dy}{dx}\,\frac{\lambda-1+(2-\lambda-\nu)x}{x(x-1)} + y\,\frac{(1-\lambda-\nu)^2-\mu^2}{4x(x-1)} = 0$$

Since the numerator of the last coefficient becomes divisible by $x(x-1)$.

For the point $x = o$, the indicial equation is now :

$$(5)\qquad r^2 - r\lambda = 0$$

of which the roots are 0 and λ. Hence the integrals in the region of the point $x = o$ are of the form

$$x^{\lambda}(e_0 + e_1 x + e_2 x^2 + \cdots\cdots) \quad \text{and}$$

$$a_0 + a_1 x + a_2 x^2 + \cdots\cdots .$$

Substituting the second series, we find by equating to o the coefficient of x^n

$$e_{n+1} = - e_n\,\frac{n^2 + n(1-\lambda-\nu) + \frac{1}{4}(1-\lambda-\nu+\mu)(1-\lambda-\nu-\mu)}{(n+1)(\lambda-1-n)}$$

making

$$1-\lambda-\mu+\nu = 2a$$
$$1-\lambda-\mu-\nu = 2b$$
$$1-\lambda \qquad = c$$

this relation becomes $e_{k+1} = e_k\,\dfrac{(a+n)(b+n)}{(n+1)(c+n)}$

and the integral is, by making $e_0 = 1$,

$$1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{1\cdot 2\, c(c+1)}x^2 + \cdots = F(a, b, c, x),$$ where

F (a,b,c,x) denotes the hypergeometric series.

Likewise by substituting the first series, we find

$$c_{n+1} = c_n \frac{\left(n + \frac{1+\lambda-\nu-\mu}{2}\right)\left(n + \frac{1+\lambda-\nu+\mu}{2}\right)}{(1+n)(n+1+\lambda)}$$

wherein, if $\dfrac{1+\lambda-\nu+\mu}{2} = a'$

$$\frac{1+\lambda-\nu-\mu}{2} = b', \quad 1+\lambda = c'$$

we obtain the relation

$$C_{n+1} = C_n \frac{(a'+n)(b'+n)}{(1+n)(c'+n)} .$$

Hence, the second integral in the region of the point $x = 0$ is $x^{\lambda} F(a', b'; c'; x)$

we conclude finally, that, in the region of the point 0,

$$P_1(\lambda, \mu, \nu, x) = C_1 F_1\left(\frac{1-\lambda-\nu+\mu}{2}, \frac{1-\lambda-\nu-\mu}{2}, 1-\lambda, x\right) + C_2 x^{\lambda} F_1\left(\frac{1+\lambda-\nu+\mu}{2}, \frac{1+\lambda-\nu-\mu}{2}, 1+\lambda, x\right)$$

where C_1 and C_2 are arbitrary constants.

Autobiography.

I, Charles Hiram Chapman, was born Oct. 25, 1859 in Columbia County, Wisconsin. When I was eight years old my parents removed to Viroqua Wisconsin, where they have since lived.

I attended the public schools in Viroqua until I was fourteen and then, after teaching one term of district school, I entered Galesville University, Trempealeau Co., Wis., where I staid a year, studying principally Latin and Greek.

From that time until the fall of 1880 I taught school in various parts of Wisconsin, Nebraska and Minnesota, occupying my leisure hours by the study of Mathematics.

In the fall of 1880 I entered the Normal School at Oshkosh, Wisconsin, and graduated thence the following spring. At the beginning of the Ses-

[illegible] of [illegible]–[illegible] I entered [illegible] this
University and took the degree of Bachelor of
Arts. Since then I have pursued my studies in
Mathematics, Physics, and Astronomy in that
University